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APPLICATION OF THE OPTIMAL LINEARIZATION METHOD TO THE HEAT TRANSFER PROBLEM

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Abstract—The generation of approximate solutions for nonlinear heat conduction problem using the method of optimal linearization is considered. Examples are used to investigate the merit of this method. Radiation cooling due to arbitrary power radiation from semi-infinite solid with temperature dependent material properties is discussed also.

NOMENCLATURE

c, heat capacity;
erf x, $\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-\lambda^2} d\lambda$;
h, constant parameter;
I, quadratic error integral;
k, thermal conductivity;
m, characteristic radiative exponent;
q_1, surface temperature;
q_2, penetration depth;
T, temperature;
t, time;
x, space coordinate;
z, dimensionless surface temperature;
α, dimensionless parameter;
δ, penetration depth;
ε, difference term; given small time interval;
κ, thermal diffusivity;
λ, constant adjustable parameter;
ρ, density;
σ, dimensionless parameter;
τ, dimensionless time.

INTRODUCTION

APPROXIMATE analytical methods of solution to heat conduction problems have received considerable attention in the last few years. This attention has been due to the necessity of including material property variations into the study of heat conduction problems.

In this note the method of optimal linearization is used to solve differential equations governing the nonlinear transfer of heat. This method was first introduced by West [1] and Blaquiere [2] in order to solve ordinary differential equations in nonlinear vibration theory (see also [3]). It seems desirable to investigate its usefulness in achieving solutions to problems in nonlinear heat conduction. The method is, however, equally appropriate for solving any problem governed by nonlinear diffusion-type equation. The solutions obtained with the help of this method, although not exact, are often sufficiently accurate for engineering purposes.

METHOD AND RESULTS

(a) *Temperature-dependent thermal conductivity*

The first application we will make of the optimal linearization method is to thermal conduction in a semi-infinite solid isotropic material. This solid is of uniform temperature $T = 0$ for $t < 0$, and its surface $x = 0$ undergoes a stepwise temperature change $T = T_0 =$ const. at the time $t = 0$.

The energy equation for the problem can be written in the form

$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{1}$$

subject to the following conditions

$$\begin{aligned} T(0,t) &= T_0 \\ T(x,0) &= 0. \end{aligned} \tag{2}$$

The thermal diffusivity $k(T)$ is assumed to be a linear function of temperature

$$k(T) = k_0\left(1 + \alpha\frac{T}{T_0}\right) \tag{3}$$

where k_0 and α are given constants.

Together with equation (1) consider the following equation

$$\rho c \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} \tag{4}$$

where λ is a constant adjustable parameter which has to be chosen in such a way that the linear equation (4) approximates equation (1) in optimal sense. To find this parameter we first form the difference of the equations (1) and (4):

$$\varepsilon\left(\lambda, T, \frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}\right) = \lambda\frac{\partial^2 T}{\partial x^2} - \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right]$$

$$= \lambda\frac{\partial^2 T}{\partial x^2} - k'(T)\left(\frac{\partial T}{\partial x}\right)^2 - k(T)\frac{\partial^2 T}{\partial x^2} \tag{5}$$

where

$$k'(T) = \frac{\mathrm{d}K(T)}{\mathrm{d}T}$$

and consider the integral

$$I(\lambda) = \int_{t_0}^{t_1}\int_{x_0}^{x_1} \varepsilon^2\left(\lambda, T, \frac{\partial T}{\partial x}, \frac{\partial^2 T}{\partial x^2}\right)\mathrm{d}x\mathrm{d}t \tag{6}$$

where the time and space intervals depend on the problem in consideration.

Suppose that there exists a known function

$$T = \psi(x,t) \tag{7}$$

which satisfies the boundary and initial conditions (2). Substituting (7) into (6) and performing integration the expression (7) will be the function of λ only

$$I(\lambda) = A\lambda^2 - 2\lambda(B + C) + D \tag{8}$$

where

$$\begin{aligned} A &= \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left(\frac{\partial^2 T}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t \\ B &= \int_{t_0}^{t_1}\int_{x_0}^{x_1} k'(T)\frac{\partial^2 T}{\partial x^2}\left(\frac{\partial T}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}t \\ C &= \int_{t_0}^{t_1}\int_{x_0}^{x_1} k(T)\left(\frac{\partial^2 T}{\partial x^2}\right)^2 \mathrm{d}x\mathrm{d}t \\ D &= \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left\{k'(T)\left(\frac{\partial T}{\partial x}\right)^2 + k(T)\frac{\partial^2 T}{\partial x^2}\right\}^2 \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{9}$$

The optimal value of λ may be found from the equation

$$\frac{\partial I(\lambda)}{\partial \lambda} = 0 \tag{10}$$

i.e.

$$\lambda = \frac{B + C}{A}. \tag{11}$$

Hence, the optimal value of λ depends on the form of chosen function $\psi(x,t)$ in (7), and the linear differential equation with the constant coefficients (4) should be considered "optimal" subject to $T = \psi(x,t)$. To be more specific, suppose the function (7) in the form

$$T(x,t) = T_0\left(1 - \operatorname{erf}\frac{x}{2\sqrt{(\kappa t)}}\right) \tag{12}$$

where

$$\kappa = \frac{k_0}{\rho c}. \tag{13}$$

The equation (12) represents the solution of boundary value problem (1) and (2) for $k(T) = k_0 = \text{const}$.

Substituting (3) and (12) into the first three terms of (9) we will get after integration with respect to x from $x_0 = 0$ to $x_1 = \infty$.

$$A = T_0^2 \frac{\sqrt{(2)}}{8\sqrt{(\pi)}} \phi(t)$$

$$B = \frac{T_0^2 k_0 \alpha}{3\pi\sqrt{(\pi)}} \phi(t) \tag{14}$$

$$C = \frac{T_0^2 k_0}{4\pi} \left\{ \frac{\sqrt{(2\pi)}}{2} + \alpha \left[\frac{\sqrt{(2\pi)}}{2} - 8\left(\frac{C_1}{4} + \frac{1}{12\sqrt{(\pi)}} \right) \right] \right\} \phi(t)$$

where

$$\phi(t) = \int_{t_0}^{t_1} \frac{\mathrm{d}t}{(\kappa t)^{\frac{1}{2}}} \tag{15}$$

and

$$C_1 = \int_0^\infty \operatorname{erf}(s) \exp[-2s^2]\, \mathrm{d}s$$

$$= \frac{1}{\sqrt{(2\pi)}} \tan^{-1} \frac{1}{\sqrt{(2)}} = 0{\cdot}2455405. \tag{16}$$

Hence from (11) we have

$$\lambda = k_0(1 + 0{\cdot}9076\,\alpha) \tag{17}$$

and the linear differential equation (4) is of the form

$$\rho c \frac{\partial T}{\partial t} = k_0(1 + 0{\cdot}9076\,\alpha) \frac{\partial^2 T}{\partial x^2}. \tag{18}$$

The solution of (19) subject to (2) is

$$T = T_0 \left(1 - \operatorname{erf} \frac{z}{\sqrt{(1 + 0{\cdot}9076\,\alpha)}} \right) \tag{19}$$

where

$$z = \frac{x}{2\sqrt{(\kappa t)}}. \tag{20}$$

This closed form solution is plotted in Fig. 1, and compared with the exact solution of Yang [4].

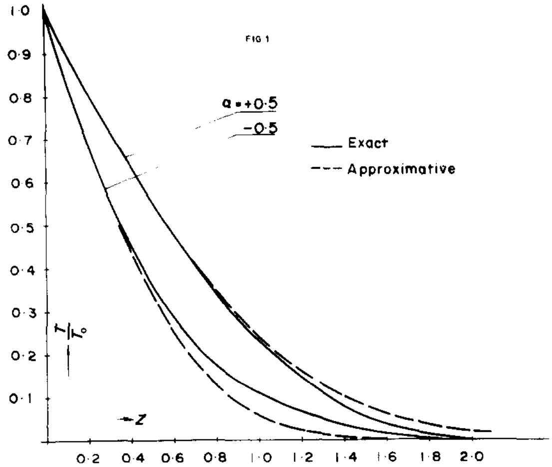

FIG. 1.

(b) *Temperature-dependent heat capacity*

Let us consider the case when the heat capacity of material is temperature dependent. The governing differential equation is

$$c(T) \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{21}$$

and initial and boundary conditions are taken to be of the same type like in case (a).

The problem is to find the best value of parameter λ that the linear equation

$$\lambda \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{22}$$

approximates in optimal way the nonlinear equation (21) together with the initial and boundary conditions. The difference term will thus be

$$\varepsilon = \lambda \left(\frac{\partial T}{\partial t} \right) - c(T) \frac{\partial T}{\partial t}. \tag{23}$$

As before, forming the mean square of the difference term (6) we will have:

$$I = \lambda^2 A - 2\lambda B + C \tag{24}$$

where

$$A = \int_{t_0}^{t_1}\int_{x_0}^{x_1} \left(\frac{\partial T}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}t$$

$$B = \int_{t_0}^{t_1}\int_{x_0}^{x_1} c(T)\left(\frac{\partial T}{\partial t}\right)^2 \mathrm{d}x\mathrm{d}t \tag{25}$$

$$C = \int_{t_0}^{t_1}\int_{x_0}^{x_1} \left[C(T)\frac{\partial T}{\partial t}\right]^2 \mathrm{d}x\mathrm{d}t.$$

Let us suppose that the heat capacity is of the form

$$c = c_0\left(1 + \frac{T}{T_0}\right) \tag{26}$$

where c_0 and T_0 are given constants and suppose that the temperature distribution (7) is of the form

$$T = T_0\left(1 - \frac{x}{a\sqrt{(\kappa t)}}\right)^2 \tag{27}$$

where

$$a = 3{\cdot}16 \tag{28}$$

and $\kappa = k/c_0$.

The expression (27) is the approximate solution of equation (21) for $c(T) = c_0 = \text{const}$, obtained by the help of variational method [5]. The term $a\sqrt{(\kappa t)} = \delta$ is the penetration depth.

Substituting (27) and (26) into (25) and performing integration with respect to x from zero to $\delta = a\sqrt{(\kappa t)}$ the expression (24) becomes

$$I = E(\tfrac{1}{3}\lambda^2 - \tfrac{6}{7}c_0\lambda + D)\,\phi_1(t) \tag{29}$$

where E and D are some constants and

$$\phi_1(t) = \int_{t_0}^{t_1} \frac{\mathrm{d}t}{t^{\frac{1}{2}}}.$$

From the condition

$$\frac{\partial I}{\partial \lambda} = 0$$

we have

$$\lambda = \tfrac{9}{7}c_0. \tag{30}$$

Hence, the linear equation (22) is of the form

$$\frac{\partial T}{\partial t} = \kappa' \frac{\partial^2 T}{\partial x^2}$$

where

$$\kappa' = \tfrac{7}{9}\frac{k}{c_0}.$$

However, the solution of this equation, according to (27) is

$$T = T_0\left(1 - \frac{x}{a\sqrt{(\kappa' t)}}\right)^2$$

and the corresponding penetration depth is

$$\delta = a\sqrt{(\kappa' t)} = 2{\cdot}78\sqrt{(\kappa t)}. \tag{31}$$

The same problem was solved by variational techniques in [5] and [6]. The corresponding values for penetration depth using quadratic profiles are

$$\delta = 2{\cdot}80\sqrt{(\kappa t)} \quad [5]$$

$$\delta = 2{\cdot}97\sqrt{(\kappa t)} \quad [6]$$

it is seen that there is quite satisfactory correlation between all three techniques over complete range x and t.

(c) *Surface radiation with variable thermal properties—short time solution*

The method of optimal linearization can be used in more complex situations in which the boundary conditions as well as the transport equation are both nonlinear. This section will treat the case where the flux condition depends upon a power of the surface temperature and at the same time, the heat capacity is a linear function of temperature. The case with constant thermal properties was discussed by Lardner in [7], using variational method. The Lardner solution will be used as the trial solution in the case of variable thermal properties.

Let us consider the surface radiation cooling of a semi-infinite solid. The initial temperature is T_0 while the ambient temperature is absolute zero. The problem we wish to solve is to find the approximate solution of partial differential equation

$$c(T)\frac{\partial T}{\partial t} = k_0 \frac{\partial^2 T}{\partial x^2} \tag{31'}$$

together with the nonlinear boundary condition

$$k_0 \frac{\partial T}{\partial x} = -hT^m \text{ on the surface } x = 0 \tag{32}$$

where

$$c(T) = c_0\left(1 + \sigma\frac{T}{T_0}\right) \tag{33}$$

and m, c_0, σ, k_0 and h are given constants.

Let us find the constant parameter λ in such a way that the linear equation

$$\lambda\frac{\partial T}{\partial t} = k_0 \frac{\partial^2 T}{\partial x^2}$$

approximates in optimal way the nonlinear equation (31′) together with the nonlinear boundary condition (32).

Substituting (34) into (25) and (24), performing integration with respect to x from zero to q_2 and with respect to t from zero to ε (ε is the range of the small time interval for which the solution (34) is valid), the condition $\partial I/\partial\lambda = 0$ yields

$$\lambda = c_0\left[1 + \sigma\left(1 - \frac{29}{112}\frac{bhT_0^{m-1}}{\sqrt{(k_0c_0)}}\sqrt{(\varepsilon)}\right)\right]. \tag{37}$$

Hence, the temperature distribution is of the form:

$$T = T_0 - \frac{bhT_0^m}{\sqrt{\left\{c_0k_0\left[1 + \sigma\left(1 - \frac{29}{112}\frac{bhT_0^{m-1}\sqrt{(\varepsilon)}}{\sqrt{(k_0c_0}}\right)\right]\right\}}}\sqrt{(t)} \times \left(1 - \frac{x}{\sqrt{\left\{\frac{k_0t}{c_0\left[1 + \sigma\left(1 - \frac{29}{112}\frac{bhT_0^{m-1}}{\sqrt{(k_0c_0)}}\sqrt{(\varepsilon)}\right)\right]}\right\}}}\right)^2. \tag{38}$$

The Lardner asymptotic solution for $\sigma = 0$ and $t \to 0$ has the form

$$T = T_0 - (T_0 - q_1)\left(1 - \frac{x}{q_2}\right)^2$$
$$= T_0 - \frac{bhT_0^m}{\sqrt{(c_0k_0)}}\sqrt{(t)}\left(1 - \frac{x}{a\sqrt{\left(\frac{k_0}{c_0}t\right)}}\right)^2 \tag{34}$$

where

$$q_1 = T_0 - \frac{bhT_0^m}{\sqrt{(c_0k_0)}}\sqrt{(t)}; \quad b = 1{\cdot}120 \tag{35}$$

is the surface temperature, and

$$q_2 = a\sqrt{\left(\frac{k_0}{c_0}t\right)}; \quad a = 2{\cdot}68 \tag{36}$$

is the depth of penetration.

The penetration distance and surface temperature are of the form

$$y = \frac{a\sqrt{(\tau)}}{\sqrt{\left\{1 + \sigma\left[1 - \frac{29}{112}\sqrt{(\tau_1)}\right]\right\}}} \tag{39}$$

$$z = 1 - \frac{b\sqrt{(\tau)}}{\sqrt{\left\{1 + \sigma\left[1 - \frac{29}{112}\sqrt{(\tau_1)}\right]\right\}}} \tag{40}$$

where

$$z = q_1/T_0$$

$$\tau = \frac{h^2}{c_0k_0}T_0^{2(m-1)}t; \quad \tau_1 = \frac{h^2}{c_0k_0}T_0^{2(m-1)}\varepsilon \tag{41}$$

$$y = \frac{h}{k_0}T_0^{2(m-1)}q_2.$$

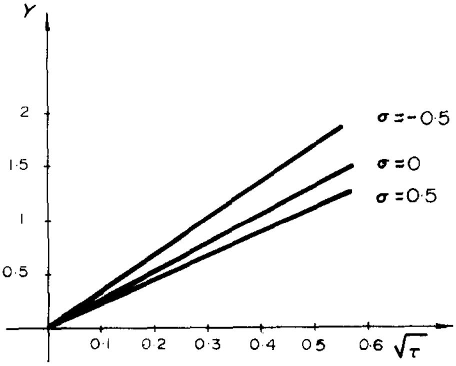

FIG. 2.

For $\sigma = 0$ the solution (39) and (40) are identical with the asymptotic short time solution given by Lardner. Figures 2 and 3 present the surface temperature and penetration depth for $\sigma = \pm 0{\cdot}5$ and $\tau_1 \approx 0{\cdot}5$. It is interesting to note that the cooling of the semi-infinite slab proceeds more slowly in the case $\sigma > 0$. For the case $\sigma < 0$ the situation is opposite. Unfortunately

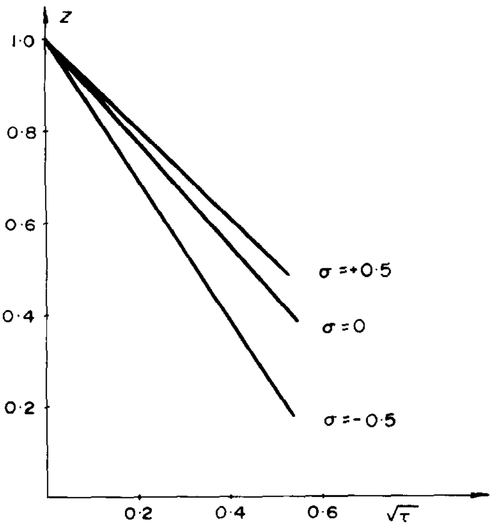

FIG. 3.

this problem is not capable of an exact solution and direct comparison is impossible.

REMARKS

The primary aim of this paper has been to demonstrate that the method of optimal linearization can be advantageously applied to the heat transfer problem. This method reduces the nonlinear boundary value problem to a linear boundary value problem whose solution can frequently be expressed in closed analytical form. The results have been presented in graphical form and comparisons have been made with other approximate solutions whenever they are available. Agreement in all cases was found to be good.

In addition to the accuracy, which is most important in any approximate solution, the method of optimal linearization has been shown to provide a systematic means of deducing the temperature history.

On the basis of the examples considered here it must be concluded that the method of optimal linearization can serve as a useful vehicle for obtaining approximate solutions in heat transfer analysis.

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APPLICATION DE LA METHODE DE LINEARISATION OPTIMALE AU PROBLEME DU TRANSFERT THERMIQUE

Résumé—On considère la génération de solutions approchées du problème de conduction thermique non linéaire par la méthode de linéarisation optimale. Dans quelques exemples, on recherche l'intérêt de cette méthode. On discute aussi le refroidissement par un rayonnement, à puissance arbitraire, d'un solide semi-infini à propriétés dépendantes de la température.

ANWENDUNG DER OPTIMALEN LINEARISIERUNGSMETHODE AUF DAS WÄRMEÜBERTRAGUNGSPROBLEM

Zusammenfassung—Es wird über die Erzeugung von Näherungslösungen für das nichtlineare Wärmeleitungsproblem unter Verwendung der Methode der optimalen Linearisation berichtet. Anhand von Beispielen wird untersucht, welchen Vorteil diese Methode bietet. Diskutiert wird ferner der Fall der Strahlungskühlung infolge Abstrahlung von einem halbunendlichen Körper unter Berücksichtigung temperaturabhängiger Materialeigenschaften.

ПРИМЕНЕНИЕ МЕТОДА ОПТИМАЛЬНОЙ ЛИНЕАРИЗАЦИИ К ПРОБЛЕМЕ ТЕПЛОПЕРЕНОСА

Аннотация—С помощью метода оптимальной линеаризации строятся приблпженные решения задачи нелинейной теплопроводности. Преимущества данного метода продемонстрированы на примерах. Также обсуждается лучистое охлаждение полубесконечного твердого тела, характеристики материала которого зависят от температуры, а мощность излучения может быть произвольной.